UNIT-IV CALCULAS

ESSAY TYPE QUESTIONS

1) Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in the interval $(-3,0)$

(i) since $x(x+3)$ being a polynomial is continuous for all values of x and $e^{-x/2}$ is also continuous for all x, their product $x(x+3)e^{-x/2} = f(x)$ is also continuous for every value of x and in particular $f(x)$ is continuous in the closed interval $[-3,0]$

(ii) we have $f'(x) = x(x+3)\left(\frac{-1}{2}e^{-x/2}\right) + (2x+3)e^{-x/2}$

$$= e^{-x/2}\left[2x+3 - \frac{x^2+3x}{2}\right] = e^{-x/2}\left[\frac{6+x-x^2}{2}\right]$$

since $f'(x)$ does not become infinite (or) indeterminate at any point of the interval $(-3,0)$ therefore, $f(x)$ is derivable in the open interval $(-3,0)$

(iii) Also we have $f(-3)=0$ and $f(0)=0$.

$$f(-3) = -3(-3+3)e^{-(-3)/2}$$

$$= -3(0)e^{3/2}$$

$$= 0$$

$$f(0) = 0(0+3)e^{0/2}$$

$$= 0$$

$$f(-3) = f(0)$$

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in the interval $[-3,0]$ Hence there exist atleast one value c of x in the interval $(-3,0)$ such that $f'(c)=0$

$$f'(c) = e^{-x/2}\left[\frac{6+x-x^2}{2}\right]$$

$$f'(c) = e^{-c/2}\left[\frac{6+c-c^2}{2}\right]$$

$$f'(c)=0$$

$$e^{-c/2}\left[\frac{6+c-c^2}{2}\right]=0$$

$$6+c-c^2=0$$

$$c^2-c-6=0$$

$$c^2-3c+2c-6=0$$

$$c(c-3)+2(c-3)=0$$

$$(c+2)(c-3)=0$$

$$c=3,-2$$

clearly the value $c=-2$ lies with in the open interval $(-3,0)$ which verifies Rolle's theorem

2) Using Mean value theorem prove that $\tan x > x$ in $0<x<\pi/2$

Consider $f(x)=\tan x$ in $0<x<\pi/2$

$f(x)=\tan x$ in $[\varepsilon, x]$ where $0<\varepsilon<x<\pi/2$

Apply lagranges mean value theorem to $f(x)$.

Two conditions are satisfied then there exists a point c in $0<\varepsilon<c<x<\pi/2$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\sec^2 c=\frac{\tan x-\tan\varepsilon}{x-\varepsilon}$$

$\tan x - \tan \varepsilon = (x-\varepsilon)\sec^2 c$

take $\varepsilon \to 0+0$

then $\tan x = x\sec^2 c$

But $\sec^2 c > 1$ Hence $\tan x > x$.

3) If $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{\sqrt{x}}$ prove that 'c' of the cauchy's generalized mean value theorem is the geometric mean of 'a' and 'b' for any $a>0, b>0$

i) clearly f, g are continuous on $[a,b] \subseteq R^+$

ii) $f'(x)=\frac{1}{2\sqrt{x}}$ and $g'(x)=\frac{-1}{2x\sqrt{x}}$ which exist on (a,b)

f, g are differentiable on $(a,b) \subseteq R^+$

iii) $g'(x)=\frac{-1}{2x\sqrt{x}} \neq 0 \ \forall x \in (a,b) \subseteq R^+$.

Thus the conditions of cauchy's mean value theorem are satisfied on (a,b)

$\therefore$ Then there exists $c \in (a,b) \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\frac{1}{\cancel{2}\cancel{\sqrt{c}}} \times \frac{-\cancel{2}c\cancel{\sqrt{c}}}{1} = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{b}\sqrt{a}}}$$

$$+c = \frac{\sqrt{b}-\sqrt{a}}{+(\sqrt{b}-\sqrt{a})} \times \sqrt{b}\sqrt{a}$$

$$c = \frac{\cancel{\sqrt{b}-\sqrt{a}}}{\cancel{\sqrt{b}-\sqrt{a}}} \times \sqrt{b}\sqrt{a}$$

$$c = \sqrt{b}\sqrt{a}$$

$$c = \sqrt{ab}$$

since $a,b>0$, $\sqrt{ab}$ is their geometric mean and we have $a<\sqrt{ab}<b$

$\therefore c\in(a,b)$ which verifies cauchy's mean value theorem

4) Find the region in which $f(x)=1-4x-x^2$ is increasing and the region in which it decreasing using Mean value theorem.

Given $f(x)=1-4x-x^2$

$f(x)$ being a polynomial function is continuous on $[a,b]$ and differentiable on (a,b) for $a,b\in R$

$\therefore f$ satisfies the conditions of lagrange mean value theorem on every interval on the real line

$$f'(x) = -4-2x = -2(2+x) \text{ for all } x\in R.$$

and $f'(x)=0$ if $x=-2$

for $x<-2$, $f'(x)>0$ and for $x>-2$, $f'(x)<0$

Hence $f(x)$ is strictly increasing on $(-\infty,-2)$ and strictly decreasing on $(-2,\infty)$

(5) Prove that $\frac{\pi}{6}+\frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6}+\frac{1}{8}$

Let $f(x)=\sin^{-1}x$, $\sin^{-1}x$ is continuous and differentiable in $[0,\pi]$.

Then $f'(x)=\frac{1}{\sqrt{1-x^2}}$

By Lagrange's mean value theorem, there exists a point c in $(0,\pi)$ such that $0<c<\pi$ and

$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{1}{\sqrt{1-c^2}}$$

Now $a<c<b$

$\Rightarrow a^2<c^2<b^2$

$\Rightarrow 1-a^2>1-c^2>1-b^2$

$\Rightarrow \sqrt{1-a^2}>\sqrt{1-c^2}>\sqrt{1-b^2}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}}<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}}<\frac{f(b)-f(a)}{b-a}<\frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}}<\frac{\sin^{-1}(b)-\sin^{-1}(a)}{b-a}<\frac{1}{\sqrt{1-b^2}}$ $\qquad [\because f(b)=\sin^{-1}b,\ f(a)=\sin^{-1}a]$

$\Rightarrow \frac{b-a}{\sqrt{1-a^2}}<(\sin^{-1}(b)-\sin^{-1}a)<\frac{b-a}{\sqrt{1-b^2}}$

Put $a=1/2$ and $b=3/5$. Then

$$\frac{(3/5-1/2)}{\sqrt{(1-1/4)}}<(\sin^{-1}(3/5)-\sin^{-1}(1/2))<\frac{(3/5-1/2)}{\sqrt{1-(3/5)^2}}$$

$\Rightarrow \frac{2}{10\sqrt{3}}<\left(\sin^{-1}\frac{3}{4}-\frac{\pi}{6}\right)<\frac{1}{10\cdot\left(\frac{4}{5}\right)}$

$\Rightarrow \frac{\pi}{6}+\frac{1}{5\sqrt{3}}<\sin^{-1}\frac{3}{5}<\frac{\pi}{6}+\frac{1}{8}$

6) Verify Rolle's theorem for the function $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$, $a>0$, $b>0$

Let $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right] = \log(x^2+ab) - \log x - \log(a+b)$

i) Since $f(x)$ is a composite function of continuous functions in $[a,b]$,

it is continuous in $[a,b]$

ii) $f'(x) = \frac{1}{x^2+ab} 2x - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$

$\therefore f'(x)$ exists for all $x \in (a,b)$

iii) $f(a) = \log\left[\frac{a^2+ab}{a^2+ab}\right] = \log 1 = 0$

$f(b) = \log\left[\frac{b^2+ab}{b^2+ab}\right] = \log 1 = 0$

$\therefore f(a) = f(b)$

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem

$\therefore$ There exists $c \in (a,b)$ such that $f'(c) = 0$

$\Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0$

$\Rightarrow c^2 = ab$

$\Rightarrow c = \pm\sqrt{ab}$

$\therefore c = \sqrt{ab} \in (a,b)$

Hence Rolle's theorem is verified.

7 Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 2 terms in the interval $[0,1]$

Consider $f(x) = (1-x)^{5/2}$ in $[0,1]$

i) $f(x), f'(x)$ are continuous in $[0,1]$

ii) $f''(x)$ is differentiable in $(0,1)$

Thus $f(x)$ satisfies the conditions of Taylor's theorem

we consider Taylor's theorem with Lagrange's form of remainder

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x) \text{ with } 0<\theta<1 \quad -(1)$$

Here $n=p=2$, $a=0$ and $x=1$

$f(x) = (1-x)^{5/2} \Rightarrow f(0)=1$

$f'(x) = \frac{-5}{2}(1-x)^{3/2} \Rightarrow f'(0) = \frac{-5}{2}$

$f''(x) = \frac{15}{4}(1-x)^{1/2} \Rightarrow f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2} \Rightarrow f''(\theta) = \frac{15}{4}(1-\theta)^{1/2}$

and $f(1)=0$

from (1), we have $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x)$

Substituting the above values, we get

$$0 = 1 + 1\left(\frac{-5}{2}\right) + \frac{1^2}{2!} \cdot \frac{15}{4}(1-\theta)^{1/2}$$

$$\Rightarrow \theta = \frac{9}{25} = 0.36$$

$\therefore \theta$ lies between 0 and 1

Thus the Taylor's theorem is verified.

Q. Show that $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + \cdots$

Let $f(x) = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$. Then $f(0)=0$

$\Rightarrow \sqrt{(1-x^2)}\, f(x) = \sin^{-1}x \quad - (1)$

Differentiating (1) w.r.t. 'x', we get

$\sqrt{1-x^2}\, f'(x) + f(x)\cdot\frac{-2x}{2\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$, using product rule

$\Rightarrow (1-x^2) f'(x) - x f(x) = 1 \quad - (2)$

Now $f'(0)=1$

Diff. (2) w.r.t. 'x', we get

$(1-x^2) f''(x) + f'(x)(-2x) - x f'(x) - f(x) = 0 \quad - (3)$

$\Rightarrow (1-x^2) f''(x) - 3x f'(x) - f(x) = 0$

$\Rightarrow f''(0) - f(0) = 0 \Rightarrow f''(0) = 0$

Diff. (3) w.r.t. 'x', we get

$(1-x^2) f'''(x) - 2x f''(x) - 3f'(x) - 3x f''(x) - f'(x) = 0$

$\Rightarrow (1-x^2) f'''(x) - 5x f''(x) - 4f'(x) = 0$

$\Rightarrow f'''(0) - 4f'(0) = 0 \Rightarrow f'''(0) = 4$

Similarly, $f^4(0) = 0$

we have by Taylor's theorem,

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^4(0) + \cdots$

i.e., $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = 0 + 1\cdot x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\cdot 4 + \cdots = x + 4\cdot\frac{x^3}{3!} + \cdots \infty$

9) Find the volume of the solid generated by the revolution of the area bounded by $y=x^2$ and $y=x$ about y-axis.

Solution:-

Given curves are:-

$$y=x^2 \longrightarrow (1)$$

$$y=x \longrightarrow (2)$$

To find the points of intersection of the given curves, solve (1) and (2).

Substituting (2) in (1), we get.

$x^2-x=0$ i.e., $x(x-1)=0$

$\therefore x=0$ (or) $x=1$. Thus $y=0$; $y=1$ [using (2)]

Hence ~~the~~ points of intersection of the two curves are $(0,0)$ and $(1,1)$.

$\therefore$ Required volume $= \pi \int_0^1 (x_2^2 - x_1^2)\, dy$

$$= \pi \int_0^1 [(x \text{ of upper curve})^2 - (x \text{ of lower curve})^2]\, dy$$

$$= \pi \int_0^1 [(\sqrt{y})^2 - (y)^2]\, dy.$$

$$= \pi \int_0^1 (y - y^2)\, dy$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$$

$$= \pi \left[\frac{1}{2} - \frac{1}{3} \right] = \pi/6$$

10, Find the Volume of the solid when Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(0<b<a)$ Rotating about minor axis.

Solution:-

Equation of the Ellipse is

$$\frac{x^2}{a^2}+y^2/b^2=1$$

$$x^2/a^2=1-y^2/b^2$$

$$x^2/a^2=\frac{b^2-y^2}{b^2}$$

$$x^2=\frac{a^2}{b^2}[b^2-y^2]$$

$\therefore$ Required Volume $=\int_{-b}^{b}\pi x^2 dy=\frac{\pi a^2}{b^2}\int_{-b}^{b}(b^2-y^2)\cdot dy$

$$=\frac{2\pi a^2}{b^2}\int_0^b (b^2-y^2)dy$$ [$\because$ Integrated in even function].

$$=\frac{2\pi a^2}{b^2}\left[b^2y-\frac{y^3}{3}\right]_0^b$$

$$=\frac{2\pi a^2}{b^2}\left[b^3-\frac{b^3}{3}\right]$$

$$=\frac{2\pi a^2}{b^2}\left[b^3-\frac{b^3}{3}\right]$$

$$=\frac{2\pi a^2}{b^2}\cdot\frac{2b^3}{3}=\frac{4\pi a^2 b}{3}$$ cubic units,

11) Find the Volume of the solid generated by revolving the $x^2/a^2 + y^2/b^2 = 1$ $(0 < b < a)$ (or) $(a > b)$ about major axis.

Solution:-

Given Equation of the ellipse is $x^2/a^2 + y^2/b^2 = 1$

when $y = 0$; $x = \pm a$

$\therefore$ Major axis of the ellipse is $x = -a$ to $x = +a$.

$\therefore$ The Volume of the solid generated by the given ellipse revolving about the major axis $= \int_{-a}^{a} \pi y^2 . dx = 2\pi \int_0^a y^2 dx$

$$= 2\pi \int_0^a \left[b^2 - \frac{b^2}{a^2} x^2 \right] . dx = 2\pi \left[b^2 x - \frac{b^2}{a^2} . \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[b^2 a - \frac{b^2}{a^2} . \frac{a^3}{3} - 0 \right]$$

$$= 2\pi \left[ab^2 - \frac{ab^2}{3} \right]$$

$$= \frac{4}{3} \pi a b^2$$

12) Show that $\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(x+a)^{m+n}} = \frac{\beta(m,n)}{a^n (1+a)^m}$

Solution:-

Proof, By definition, We have $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \longrightarrow ①$

put $x = \dfrac{(1+a)t}{t+a}$.

then $\dfrac{dx}{dt} = (1+a)\left[\dfrac{(t+a).1 - t(1+a)}{(t+a)^2}\right]$

$$\frac{dx}{dt} = \frac{a(1+a)}{(t+a)^2} \quad (\text{or}) \quad dx = \frac{a(1+a)}{(t+a)^2}.dt$$

Also when $x=0, t=0$ and when $x=1, t=1$.

Now (1) becomes.

$$B(m,n) = \int_0^1 \frac{(1+a)^{m-1} t^{m-1}}{(t+a)^{m-1}} \left[1 - \frac{(1+a)t}{t+a}\right]^{n-1} \cdot \frac{a(1+a)}{(t+a)^2}.dt$$

$$= \int_0^1 \frac{(1+a)^m t^{m-1}}{(t+a)^{m+1}} \cdot \left[\frac{a-at}{t+a}\right]^{n-1} a\,dt$$

$$= \int_0^1 \frac{a^n (1+a)^m t^{m-1} (1-t)^{n-1}}{(t+a)^{m+n}}.dt$$

$$= a^n (1+a)^m \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(a+t)^{m+n}}.dt$$

$$= a^n (1+a)^m \int_0^1 \frac{a^n (1+a)^m t^{m-1} (1-t)^{n-1}}{(t+a)^{m+n}}.dt$$

$$= a^n (1+a)^m \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}.dx$$

$$\therefore \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}.dx = \frac{B(m,n)}{a^n (1+a)^m}$$

13 Prove that $p(m,n) = \dfrac{\sqrt{(m)}\cdot\sqrt{(n)}}{\sqrt{m+n}}$

proof : from definition, we have $\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx$ ---- ①

put $x = yt$ so that $dx = y$ then (1) gives

$$\Gamma(m) = \int_0^\infty e^{-yt}\, y^{m-1} t^{m-1}\, y\, dt$$

$$= \int_0^\infty y^m e^{-yt} t^{m-1} dt$$

$$= \int_0^\infty y^m e^{-yx} x^{m-1} dx \quad \cdots (2)$$

or $\dfrac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx} x^{m-1} dx$ ---- (3)

multiplying both side (3) by $e^{-y} \cdot y^{m+n-1}$, we get

$$\Gamma(m)\int_0^\infty e^{-y} y^{n-1} dy = \int_0^\infty \left\{\int_0^\infty e^{-y(1+x)} y^{m+n-1} x^{m-1} dx\right\} dy \quad \text{---(4)}$$

Integrating both side of (4) w.r.t y form 0 to ∞, we have

$$\Gamma(m)\int_0^\infty e^{-y} y^{n-1} dy = \int_0^\infty \left\{\int_0^\infty e^{-y(1+x)} y^{m+n-1} x^{m-1} dx\right\} dy$$

Or $\Gamma(m)\Gamma(n) = \int_0^\infty \left\{\int_0^\infty e^{-y(1+x)} y^{m+n-1} dy\right\} x^{m-1} dx$, by interchanging the order of integration

$$\therefore \Gamma(m)\Gamma(n) = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1} dx, \text{ by } (3)$$

$$= \Gamma(m+n) \cdot \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$= \Gamma(m+n) B(m,n)$ [by form II of Beta function]

Hence $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

14) Express $\int_0^1 x^m(1-x^n)^p dx$ in terms of Γ functions and hence evaluate $\int_0^1 x^5(1-x^3)^{10}dx$.

Sol:- put $x^n = y \Rightarrow x = y^{1/n}$

So that $dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$

$$\therefore \int_0^1 x^m(1-x^n)^p dx$$

$$= \int_0^1 y^{m/n}(1-y)^p \frac{1}{n} y^{\frac{1-n}{n}} dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m+1-n}{n}} (1-y)^p dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m+1}{n}-1} (1-y)^{(p+1)} dy$$

$$= \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right)$$

$$= \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

$$\left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

Deduction : $m=5$, $n=3$ and $p=10$ in the above result.

(15) Evaluate the following

i) $\int_0^1 x^4 (\log 1/x)^3 dx$

put $\log \frac{1}{x} = t$ i.e, $\frac{1}{x} = e^t$ (or) $x = e^{-t}$

$\therefore dx = -e^{-t} dt$

Also when $x = 1$, $t = 0$ and when $x \to 0$, $t \to \infty$

$$\therefore \int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx$$

$$= \int_\infty^0 e^{-4t} t^3 . (-e^{-t} dt)$$

$$= \int_0^\infty e^{-5t} t^3 dt$$

put $5t = u$ so that $dt = \frac{du}{5}$

$$\therefore \int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx = \int_0^\infty e^{-u} \left(\frac{u}{5}\right)^3 \frac{du}{5}$$

$$= \frac{1}{625} \int_0^\infty e^{-u} u^3 du$$

$$= \frac{1}{625} \int_0^\infty e^{-u} u^{4-1} du$$

$$= \frac{1}{625} \cdot \Gamma(4) = \frac{3!}{625}$$

$$= 6/625$$

ii) $\int_0^\infty e^{-x^2}dx$

$a^2x^2 = y \Rightarrow x = \frac{\sqrt{y}}{a}$

so that $dx = \frac{1}{2a} y^{-1/2} dy$

$\therefore \int_0^\infty e^{-a^2x^2} dx$

$= \int_0^\infty e^{-y} \frac{1}{2a} y^{-1/2} dy$

$= \frac{1}{2a} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2a}$

put $a^2 = 1$ then $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

iii) $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$

put $x^2 = y \Rightarrow x = y^{1/2}$ so that $dx = \frac{1}{2} y^{-1/2} dy$

$\therefore \int_0^\infty \sqrt{x}\, e^{-x^2} dx = \int_0^\infty y^{1/4} e^{-y} . \frac{1}{2} y^{-1/2} dy$

$= \frac{1}{2} \int_0^\infty e^{-y} y^{-1/4} dy$

$= \frac{1}{2} \int_0^\infty e^{-y} y^{\frac{3}{4}-1} dy$

$= \frac{1}{2} \Gamma\left(\frac{3}{4}\right)$

iv) $\int_0^{\pi/2} \sin^2\theta \cdot \cos^4\theta \, d\theta$

Sol:- we have $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$ ---- ①

putting $2m-1 = 2$ and $2n-1=4$ so that $m = \frac{3}{2}$ and $n = \frac{5}{2}$

then (1) becomes

$$\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta$$

$$= \frac{1}{2} B\left(\frac{3}{2}, \frac{5}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left[\frac{3}{2}\right]\Gamma\left(\frac{5}{2}\right]}{\Gamma\left(\frac{3}{2}+\frac{5}{2}\right)} \qquad \left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{2} \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)}$$

$$= \frac{3\pi}{16(3!)} = \frac{\pi}{32} \qquad [\because \Gamma(n) = (n-1)!]$$

v) $\int_0^1 \frac{x^3}{\sqrt{1-x}} \, dx.$

Sol. $\int_0^1 x^3 (1-x)^{-1/2} dx. = \int_0^1 x^{4-1} (1-x)^{1/2 - 1} dx.$

$$= \beta(4, 1/2)$$

$$= \frac{\Gamma(4)\,\Gamma(1/2)}{\Gamma(4+1/2)} = \frac{3!\sqrt{\pi}}{\Gamma(9/2)} = \frac{3!\sqrt{\pi}}{7/2 \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}} \qquad \left[\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{16}{105}$$

1(b) To find $4\int_0^{\infty} \frac{x^2}{1+x^4} \cdot dx$ using β-functions:-

Sol:- put $x = \sqrt{\tan\theta} \Rightarrow dx = \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta \, d\theta$

$L.L \Rightarrow x = 0 \Rightarrow \theta = 0$

$U.L \Rightarrow x = \infty \Rightarrow \theta = \pi/2$

$$\Rightarrow 4 \cdot \int_0^{\infty} \frac{x^2}{1+x^4} dx = 4 \int_0^{\pi/2} \frac{\tan\theta}{(1+\tan^2\theta)} \cdot \frac{1}{\cancel{2}\sqrt{\tan\theta}} \cancel{\sec^2\theta} \cdot d\theta \cdot$$

$$\Rightarrow 2 \int_0^{\pi/2} \sqrt{\tan\theta} \cdot d\theta \qquad [\sec^2 x \cdot \tan^2 x$$

$$\left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$\Rightarrow 2 \int_0^{\pi/2} (\sin\theta)^{1/2} (\cos\theta)^{1/2} d\theta.$$

$$\Rightarrow \text{But we know that } \int_0^{\pi/2} \sin^m x \cos^n x \, dx = \tfrac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$\text{and also } B(m,n) = \frac{\sqrt{m}\sqrt{n}}{\sqrt{m+n}}$$

$$\Rightarrow 2 \times \frac{1}{2} B \cdot \left[\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right]$$

$$\Rightarrow B\left[\frac{3}{4}, \frac{1}{4}\right].$$

$$\Rightarrow \frac{\sqrt{3/4}\sqrt{1/4}}{\sqrt{3/4+1/4}} = \frac{\sqrt{3/4}\sqrt{1/4}}{\sqrt{1}} \qquad [\because \sqrt{1} = 1]$$

$$\Rightarrow \sqrt{\frac{1}{4}} \sqrt{1 - \frac{1}{4}}$$

We know that $\sqrt{n}\sqrt{1-n} = \pi / \sin(n\pi)$.

$(n = 1/4)$

$$= \frac{\pi}{\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\,\pi$$

$$\boxed{\therefore\ 4\int_0^{\infty} \frac{x^2}{1+x^4}\,dx = \sqrt{2}\,\pi}$$

17 prove that $\int_0^1 (1-x^n)^{1/n}\,dx = \frac{1}{n}\frac{[\Gamma 1/n]^2}{2[\Gamma 2/n]}$

Sol:- put $x^n = y$, i.e; $x = y^{1/n}$ so that $dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$

$$\therefore \int_0^1 (1-x^n)^{1/n}\,dx$$

$$= \int_0^1 (1-y)^{1/n} \cdot \frac{1}{n} y^{\frac{1}{n}-1} dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{1}{n}-1}(1-y)^{(\frac{1}{n}+1)-1} dy.$$

$$= \frac{1}{n} B\left[\frac{1}{n}, \frac{1}{n}+1\right]$$

$$= \frac{1}{n}\frac{\Gamma\left[\frac{1}{n}\right]\Gamma\left(\frac{1}{n}+1\right]}{\Gamma\left(\frac{1}{n}+\frac{1}{n}+1\right)} = \frac{1}{n}\cdot\frac{\Gamma(\frac{1}{n})\frac{1}{n}\Gamma(\frac{1}{n})}{\Gamma(\frac{2}{n}+1)}$$

$$= \frac{1}{n^2}\frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{\frac{2}{n}\Gamma\left(\frac{2}{n}\right)}$$

$$= \frac{1}{n}\frac{\Gamma\left(\frac{1}{n}\right)^2}{2\Gamma\left(\frac{2}{n}\right)}$$

18). Show that $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \pi/4$

Sol:- consider $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}}$

put $x^2 = \sin\theta$, i.e., $x = \sin^{1/2}\theta$ so that $dx = \frac{1}{2}\sin^{-1/2}\theta \cdot \cos\theta \, d\theta$

$$\therefore \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \frac{1}{2}\sin^{-1/2}\theta \cdot \cos\theta \, d\theta$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta \cdot d\theta = \frac{1}{2}\cdot\frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \left[\because \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)\right\}$$

$$\Rightarrow \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} \quad \left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{4} = \frac{\Gamma\left(\frac{3}{4}\right)}{\left(\frac{5}{4}-1\right)\Gamma\left(\frac{5}{4}-1\right)} \quad \left[\because \Gamma(n) = (n-1)\Gamma(n-1)\right].$$

$$= \sqrt{\pi} \frac{\Gamma(3/4)}{\Gamma(1/4)} \quad \ldots (1).$$

Now consider $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

put $x^2 = \sin\theta$, i.e., $x = \sin^{1/2}\theta$ so that $dx = \frac{1}{2}\sin^{-1/2}\theta \cos\theta \, d\theta$.

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{1}{2} \frac{\sin^{-1/2}\theta \cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{0}\theta \, d\theta = \frac{1}{2}\cdot\frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{2} \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})} \quad \left[\because B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

$$\Rightarrow \frac{\pi}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)}$$

Hence $\displaystyle\int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}}$

$$\Rightarrow \frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)} \times \frac{\sqrt{\pi}\,\Gamma(1/4)}{4\,\Gamma(3/4)}$$

$= \pi/4$, using (1) and (2).